Alain Badiou

Infinitesimal Subversion[1]

*[**Editorial note: this manuscript file will be replaced by a definitive pdf in early 2013**]*

[118] I Support and Inoccupation

The finite – which Hegel describes as the iterative transgression of its own limit – is essentially that which allows, and thereby demands, a supplementary inscription. Thus, what is constitutive for it is the empty place where that inscription which it lacks is possible. A number x_n is that which determines 'to its right' the place of its successor: $(x_nS) \rightarrow (x_nSx_{n+1})$. To be inscribed at one of the places distributed by S is to assign to the other place [*l'autre place*] the constraining exclusivity of the blank space. The numerical effect exhausts itself in the incessant shunting along of the empty place: number is the displacement of the place where it is lacking [*où il manque*].

However, this operation presupposes a (unique) space of exercise, that is to say, an out-of-place blank [*blanc*] where the place is displaced in the retroaction of the inscribed – this is what Mallarmé characterized as the initial or solitary or more profoundly as the 'gratuitous' blankness or whiteness, since it is what is written that bestows upon it its status as place of the writing that takes place.[2]

This is why the 'potential' infinite, the indefiniteness of progression, testifies retroactively to the 'actual' infinity of its *support*.

It is possible to demonstrate this by objectifying the concept of effective procedure, or algorithm, as a mechanical procedure. The Turing machine, which accomplishes this programme, is realizable as a material assemblage, but only if – and this is the only thing that distinguishes it from a legible inscription on physical paper – one assumes that the tape which provides the support for its successive marks is infinite. Everything that is mathematically ideal about the Turing machine, everything in it that pertains to rational universality, is encapsulated in this postulation. It is the fact that the concept of 'algorithm' cannot be entirely articulated within odological space that defines, in keeping with this very impossibility, the reality of the infinity-support.

For an algorithmic sequence, the infinity-support is the non-markable unity of its space of inscription.

Let us now consider a domain of mathematical objects definable according to the construction procedures which their axioms prescribe. For [119] example, and as evoked above, the natural numbers defined through the logic of the 'successor' operation.

Let us suppose that these procedures allow us to designate a place such that none of the objects that are constructible within this domain can, on pain of contradiction, be marked within it. We will call 'infinity-point of the domain' the *supplementary* mark that conforms to the following conditions:

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[1] TN: First published as Alain Badiou, 'La Subversion infinitésimale', CpA 9.8 (summer 1968), 118-137. Translated by Robin Mackay with Ray Brassier.

[2] '*Quand s'aligna, dans un brisure, le moindre, disséminée, le hasard vaincu mot par mot, indéfectiblement le blanc revient, tout à l'heure gratuit, certain maintenant.*' TN: '[A]nd when in the lines chance appeared conquered word by word in a scattered minimum rupture, indefectibly the white [blank] returns, gratuitous before, now certain' (Stéphane Mallarmé, *Le Mystère dans les lettres,* in *Mallarmé*, ed. and trans. Anthony Hartley [London: Penguin, 1965], 204).

(a) It occupies the unoccupiable empty place.
(b) Apart from this occupation, it is governed by all the initial procedures.

Here, the infinite is the designation of a beyond proper to the algorithms of the domain: the marking of a point that is inaccessible[3] according to the algorithms themselves, but which supports their reiteration.

This infinite [*cet infini*] has a twofold relation to the procedures of construction, since only the latter allow us to determine the unoccupiable place which the former will come to occupy, while the former enables the efficacy of the latter to recommence. But the infinite is also exterior to the domain in which those procedures are exercised – this is its supplementarity – since it marks within this domain that which is averred in it only as void. We see then that the infinite closes off a domain by occluding the voids determined within it; but also that it opens up a higher domain as the first point of a second space in which the initial procedures can be exercised. This pulsation of closure and opening defines the infinity-point: it is the zero of a higher stratum.

Take for example the relation of order over the whole natural numbers. It allows us to construct the concept of a place which no number can occupy: the place of a number which would be larger than all others. This *place* is perfectly constructible, since the statement 'for all x, $x < y$' is a well-formed statement of the system, referring to a defined relation. Now, in this statement, the *variable* 'y' marks the place in question. However, no *constant* of the system, no *proper name* of a number, can occupy this place – i.e. can be substituted for the variable 'y' – without a contradiction ensuing. Although this place can be defined in terms of the procedures governing the numerical domain, it is nevertheless trans-numeric. *Every number is lacking in this place*.

Suppose now that I augment the system's alphabet with a constant, call it i (which is not the symbol of any number), whose usage I define in terms of the occupation of this transnumeric place, positing that, for *every* number n, $n < i$.

In terms of the 'normal' models of the system, it is clear that i *is not* a [120] whole number. However, if I can operate on i (i.e. calculate with i) without contradiction, in conformity with the initial procedures governing the domain – if, for example, I can define the successor of i, that is $i + 1$, or the sum $i + i$, and so on – then I can say that i is an *infinite whole number*. By which should be understood: an infinity-point relative to the structure of order over the domain of natural whole numbers.

Thus the infinity-point is the marking of something inaccessible for the domain; a marking completed by a *forcing* [*forçage*] of procedures, one that obliges them to apply to precisely that which they had excluded. Of course, this forcing entails a modification of the way in which the domain is set out, since the constructible objects in the higher domain are able to occupy places which those of the domain itself 'unoccupy'. The new space in which the procedures can be exercised is disconnected from that which preceded it. The models of the system are stratified. We will call these effects of the marking of constructible empty places a *recasting* [*refonte*].[4]

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[3] In set theory, an inaccessible cardinal is precisely an infinity-point, relative to cardinals smaller than it, for certain expansive algorithms: (*a*) passage to the set of parts, (*b*) passage to the union-set, or set of elements of all sets which are elements of the initial set.

[4] We have taken the concept of *recasting* [*refonte*] from François Regnault. He uses it to designate those great modifications whereby a science, returning to what was un-thought in its preceding epoch, carries out a global transformation of its system of concepts – *e.g.* relativistic mechanics after classical mechanics.

TN: *Refonte* can be translated as reconfiguration or overhaul, but its primary meaning – of recasting or reforging – is more evocative of the processes at work here. Regnault appears to have adapted the term from Bachelard. 'Crises in the development of thought imply a total recasting [*refonte*] of the system of knowledge. The mindset [*la tête*] must at such a time be remade. It changes species [...]. By the spiritual revolutions required by a scientific invention, man becomes a mutable species, or better, a

The infinity-point of a domain is a recasting-inscription.

Note that while the infinity-support is required by the recurrent *possibility* of inscribing a mark in the empty place assigned by the primitive relation of the domain, conversely, it is the *impossibility* of a certain mark within that domain that gives rise to the infinity-point. While the former supports the rules of construction, the latter, which is inaccessible, recasts and relaunches them, thereby determining a new space of inscription, a difference in the support: *the infinity-point is the differential of the infinity-support*.

II Variable Signature of a Real

We will now examine the following paradox: defining a concept of the infinite in terms of the inoccupation of a place, we have nonetheless conceded that in a certain sense this place was always already marked. How then are we to recognize it, if it dissipates itself in the retrospective indistinction of the infinity-support? Being obliged to write that the place is unoccupiable, no doubt I must inscribe what will attest that it is *this* place, and no other. To differentiate the unoccupiable place requires the occupation constituted by the mark of this difference.

And in fact we have consented to write, without claiming to have gone beyond what is permitted by the laws of the domain, 'for all x, $x < y$'. What about this 'y', which we call a variable, which occupies the place in which no constant can be inscribed [121], and where the supplementary symbol will only come to be inscribed by forcing the recasting of the entire domain? And if the infinity-point is only what is substituted for a variable, must we not attribute to the latter the power, internal to the domain, of occupying the empty place, such that the true concept of the infinite would already be enveloped in the mobile inscription of *x*'s and *y*'s?

This is indeed what many epistemologies declare, Hegel's included. The literal inscriptions of algebra, such as a/b, are, relative to a given quantitative domain, 'general signs' (*allgemeine Zeichen*).[5] This means: substitutive infinities, whose finitude in inscription holds and gathers the scattered virtuality of inscription of all those quanta of the domain whereby one can, in calculating, substitute for *a* or *b*. Letters here are 'indeterminate possibilities of every determinate value',[6] with the indetermination of quantitative possibility [*du possible quantitatif*] finding its fixed qualitative closure in the formal invariance of the mark – in Hegel's example, the *relation* a/b, the bar /.

What Hegel thinks in this text is the *logical* concept of the variable, because he rightly rejects the notion of a 'variable magnitude', which he considers vague and improper.[7] Indeed, the idea of the variability of a magnitude confuses functional considerations (variations of a function) with algebraic considerations (literal or undetermined symbols); it conceals *substitution* under *correlation*. Hegel prefers instead the concept of that which, although related to a quantity (to number) is not a quantum. Letters (*die Buchstaben*)[8] are variables by virtue of the proper difference which assigns them to quanta, just as in logic one distinguishes between two lists of symbols of individuals: constants, i.e. proper names for which, once inscribed, nothing can be substituted; and variables, for which under certain circumstances constants can be

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species that needs to mutate, one that suffers from failing to change' (Gaston Bachelard, *La Formation de l'esprit scientifique* [1938] [Paris: Vrin, 2004], 18; *The Formation of the Scientific Mind*, trans. Mary McAllester Jones [Manchester: Clinamen, 2002], 26tm).

5 Georg W.F. Hegel, *Science of Logic*, trans. George di Giovanni (Cambridge: Cambridge University Press, 2010), 21:243. TN: The pagination here refers to the German *Gesammelte Werke*. (There appear to be some inconsistencies in Badiou's own listing of page numbers in the French version of this article).

6 Hegel, *Science of Logic*, 21:243tm.

7 Ibid., 21:249.

8 Ibid., 21:243.

substituted. Because of this capacity to disappear to make space for the fixity of marks, variables participate in true infinity: the dialectical sublation of the infinity of iteration.[9]

And it is true that the variable appears to be a crossroads of infinities. We have just seen in what sense it harboured anticipatively the powers of the infinity-point. But insofar as it can be replaced with a constant, and is exhausted in supporting virtual substitutions, the variable seems to mark *all* the places of the domain under consideration that can be occupied by constants. Thus, the variable could index [*indexer*] the infinity-support. This is indeed how Quine understands it when he quips: 'to be is to be the value of a variable',[10] if the being in question is the materiality of the mark and the ontological site is the space of its inscription.

However, this is not at all the case. As an effective inscription, the variable presupposes the infinity-support as the site of places. Placed there where a constant can come, it belongs to the same order of markings as that constant, rather than designating its type.

No doubt the variable marks a *constructible*, albeit not necessarily [122] *occupiable*, place of the domain. But this marking is entwined with the domain's own law, with its algorithmic finitude. Even if I inscribe a variable in an unoccupiable place, I do not for all that infinitize the domain; I do not transgress its rule, having thereby merely afforded myself the means of *writing the impossibility of the impossible*.

Take for example, in the domain of whole naturals, the equation:

$$4 - x = x$$

This is a possible equation, unlike, for example, $4 - 7 = 7$, which is not merely false but is, within the domain, strictly illegible, the term $(4 - 7)$ not being well-formed.

The general (indeterminate) possibility of writing $4 - x = x$, and, let us say, $x > 4$, allows me to *state* the impossibility of their conjoint inscription, in the form of the statement [*écriture*]:

$$\text{not-}(4 - x = x \text{ and } x > 4)$$

This is a statement in which *no* constant can take the place marked by the variable x yet which, at the same time, writes this very impossibility. Here the variable enables the *explicit* marking of the unoccupiability of a constructible place.

Let us say that a variable ensures that impossible equations are sufficiently legible to read their impossibility.

Now, following a proposition of Lacan's, the real for a domain of fixed proofs is defined as what is impossible. It is by excluding certain statements, and by the impossibility for any constant of occupying certain constructible places, that an axiomatic system can operate as *this* system, and can allow itself to be thought differentially as the discourse of a real.

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[9] TN: Badiou specifies here that he has taken from Jacques Derrida the French translation of Hegel's *Aufhebung* as *relève*; we have used the now-conventional English term 'sublation'.

[10] TN: 'Whatever we say with help of names can be said in a language which shuns names altogether. To be is, purely and simply, to be the value of a variable. In terms of the categories of traditional grammar, this amounts roughly to saying that to be is to be in the range of reference of a pronoun. Pronouns are the basic media of reference; nouns might better have been named pro-pronouns. The variables of quantification, "something", "nothing", "everything", range over our whole ontology, whatever it may be; and we are convicted of a particular ontological presupposition if, and only if, the alleged presuppositum has to be reckoned among the entities over which our variables range in order to render one of our affirmations true' (Willard V. Quine, 'On What There Is', *The Review of Metaphysics* 2:5 [September 1948], 32; cf. Quine, 'Notes on Existence and Necessity', *Journal of Philosophy* 11 [1943], 113-127).

If every statement is derivable, the system is inconsistent; if every constructible place is occupiable, the system, marking neither differences nor regions, becomes an opaque body, a deregulated grammar, a discourse dense with nothingness. The variable, as inscription which disjoins the constructible from the occupiable – governing which constants belong to the former but not to the latter – testifies to the intra-systemic trace of the system's reality. The operator of the real for a domain, it in fact authorizes *within* that domain the writing of the impossible proper to it. The existent has as its category a being-able-not-to-be the value of a variable at the place it marks.

In this regard, the variable is the exact inverse of the infinity-point, whose inscription it prepares.

For it is in this place of the impossible, which the variable occupies *in order to designate its impossibility*, that the infinity-point will come to inscribe itself as a constant. The infinity-point once again occupies the unoccupiable place, it substitutes itself for the variable, but according to the writing *of the possibility of the impossible*. There is now a constant where the variable traced the prescribed lack of every constant. The infinity-point is the becoming-constant of a variable in the impossible place whose impossibility it indexes. [123]

The variable *realises* the difference of a system as the pure wake or trace [*sillage*] left by the disappearance of a mark – of a constant – whose lack-in-its-place the variable names. The infinity-point, through which this mark makes its return into the system, *irrealises* the latter: this is something mathematicians already knew, since they successively named 'irrational' and 'imaginary' those infinity-points for the domain of relations of whole numbers, and for the domain which, in the retrospection of its recasting, was constituted as 'real'.

Lacan would call this the hallucinatory position of the infinity-point, whose variable, far from enveloping its coming-forth, has instead marked its prosaic exclusion.

Thus the infinity-point, however much it may proliferate after the recasting, is axiomatically one, or a closed list [*liste close*]; whereas the variable is, one might say, as numerous as the constants: to write $x < y$ is another thing altogether than writing $x < x$, since the impossibility must be evaluated for *each* place, rather than the infinity-point relative to an algorithm being linked to *an* unoccupiable place, and the infinity-support, originally, to *every* place.

In a logical calculus, the list of variables is open. Far from folding the differences of the domain back into the unity of a mark, the variable, as instrument of the real of places, only redoubles them, distributing as many proper impossibilities as there are constants capable of entering or not entering into any given relation [*relation quelconque*].

The variable as mark is unable to figure the Infinity of marks of a domain, since it is coextensive with their reality.

III To Mark the Almost-Nothing?

We are now going to deal with a particular class of marks, which, after some initial successes, were long held to be inadmissible: infinitesimal marks, in which the impossible and the infinite, the variable and the point, are distributed in the now unravelled history of a repression.

The intrinsic absurdity of an infinitely small number was indeed the dogmatic result of a very long journey punctuated, in its speculative origins, by Zeno's paradoxes. It is no exaggeration to say that an entire mathematico-philosophical tradition is bound up in it, a secular tradition whose unity is the result of a rejection – rejection of the minimal differential element which would seem to be inscribed as such in the fabric of continuity. The very opposition between indivisible atoms and the infinite divisibility of the continuum is maintained in the unitary space of this exclusion, since the *real* indivisibility of the atom assigns it a (very small) unit of dimension, rather than a punctuality; whereas the infinite un-interruptedness of divisibility is precisely what rules out the notion of an actual infinitesimal stopping-point. [124]

Whence the fact that Hegel can simultaneously approve of the 'atomistic principle', even the atomistic mathematics apparently delineated by Bonaventura Cavalieri's *indivisibles*, and the infinite divisibility of the continuum: he perceives with acuity their dialectical correlation, whose signature is the annulment of the infinitely small as such.

With regard to Cavalieri, Hegel shows how, for example, although hampered by an inadequate language, what the Italian mathematician envisages is not a *composition* of the spatial continuum by discrete elements, but the principle of a *relation* of magnitude. The primacy of the discrete is in no way restored here. No doubt 'the image of an *aggregate* of lines is incompatible with the continuity of the figure.'[11] But Cavalieri knows this perfectly well. His conception is not set-theoretical, the continua [*les continus*] are not collections of indivisibles: 'continuous figures follow only the *proportion* of the indivisibles.'[12] We must understand that the atomism of indivisible elements serves only for the comparative ciphering of figures, leaving their continuous-being entirely untouched. 'The lines do not in fact constitute the whole content of the figure as *continuous*, but only the content in so far as it is to be arithmetically *determined*.'[13] In short: geometrical continuity is the void wherein indivisible atoms come to inscribe relations of magnitude. And this inscription does not breach the *infinite* divisibility of the continuous, a pure possibility left open by a relation of indivisibles which do not denote the former's quantitative *being*, but its figuration in the formal (qualitative) structure of this relation.[14]

The divisibility of the continuous in turn delivers no proper indivisible element. The decomposition of the continuum cannot reach an indivisible element, or even the reality of an 'infinitely small' part, any more than indivisible elements are able to compose the continuum. The division of the continuum is undone as soon it is posited, thereby restoring the adherence, the inseparable connectivity of the whole: 'Divisibility itself is only a possibility, and not a *concrete existence of parts*; multiplicity in general is only posited in continuity as a moment, immediately suppressed [or sublated].'[15]

Neither progression nor regression. In classical epistemology, we find a complicity of the atomistic and the continuous.

For as Hegel remarks,[16] the atom is never an infinitesimal of the continuum. The atom is the (arithmetical) One whose *combinatorial* proliferation produces not the continuum but the thing against the *backdrop* of the continuum. The veritable non-composable principle of the continuum and of movement remains the *void*, the unique [125] space of the inscription of Ones, the infinity-support wherein atomistic discreteness is marked. Hegel has no difficulty recognizing, in the retroactive continuity of the void, the cause of the mobile combination of atoms, the continuous restlessness of the negative, which obliges the discrete to determine itself as numeral [*numéral*], as a thing made of atoms.

Thus it appears that the couplet atoms/void, the physical objectivation of the couplet discrete/continuous, is constituted by excluding every infinitesimal composition

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[11] Hegel, *Science of Logic*, 21:305.
[12] Ibid., 21:306.
[13] Ibid., 21:305.
[14] On this point Alexandre Koyré takes up Hegel's argument, without explicitly mentioning it, in his 'Bonaventura Cavaliéri et la géométrie des continus', an article of 1954, reprinted in Koyré's *Études d'histoire de la pensée scientifique* (Paris: Gallimard 1973), 334-361.
[15] Hegel, *Science of Logic*, 21:188tm.
[16] Ibid., 21:153ff.

of the continuum itself: there may be atoms *in* the void, but there are no atoms *of* the void.

Conversely, the Euclidean definition of the magnitude of a given type prohibits any foreclosure of the process of increase-decrease whose permanent possibility *is* the very concept of magnitude: 'We say that magnitudes (μεγεθη) have a relation (λογος) between them when one of them can, when multiplied, surpass the other.'[17] From this Hegel concludes, in an accurate interpretation of the intentions of Greek mathematics, that a supposedly infinite *element* which, whether multiplied or divided, can never equal any finite magnitude whatsoever, has no relation at all with such a magnitude: 'Given that the infinitely large and the infinitely small cannot be respectively increased or diminished, neither one nor the other are, in fact, quanta'.[18] To attempt to think infinities as such, to *mark* them as numbers, amounts to establishing oneself strictly within the αλογος, the radical non-relation. One cannot therefore write an infinitesimal mark, for example *dx*, except in the composition of an *already given* relation [with *dy*], and remembering all the while 'that outside of this relation, it [the *dx*] is null'[19] – a nullity whose force is absolute, excluding any separate mention of *dx*. The *dx* is nothing, not even an acceptable symbol, outside the place assigned to it by the bar /.[20] The *dx* as mark is *adherent* to a determined blank space: it is the pre-existent bar of that relation alone that renders its inscription possible. For Hegel, it is precisely this anteriority of the bar that constitutes the quality of the differential, and thus its infinity.

Whence the obvious conclusion that in the expression 'infinitely small', 'small' means nothing, since outside the (qualitative) form of the relation, one cannot evaluate the magnitude of that which is nothing but a null mark, *dx*. Note that the same thing holds in contemporary analysis: if the separate mention of the differential is the rule there, this is not on account of its being any more of a *quantum*, but rather precisely because it is held to be an *operator*: it would thus be absurd to evaluate its magnitude.

Thus, historically, the mathematical project set about dispensing with any mention of the quantified infinite. Joseph Louis Lagrange, Hegel's principal scientific source, announces this expressly in the very title of his canonical work: *Theory of Analytical Functions, Containing the Principles of* [126] *Differential Calculus, Disengaged from all Consideration of Infinitely Small, Vanishing Quantities, Limits and Fluxions, and Reduced to the Algebraic Analysis of Finite Quantities* (1797).

The gesture of rejection is constitutive: the *impurity* of the origin of differential calculus was the isolated marking, the trace of the infinitely small. Thus the story of this calculus is also that of the effacement of this trace.

Remarkably, these conclusions were to survive, essentially intact, the Cantorian recasting, which as we know completely revolutionized the concept of the infinitely *large*. Georg Cantor displayed a truly Greek intransigence in his refusal of the infinitely *small*. As late as 1928, Abraham Fraenkel, faithfully echoing the master, writes:

> Put to the test, the infinitely small failed utterly.
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> The diverse types of infinitely small considered up to the present time, some meticulously founded, have proved totally useless for getting to the bottom of the simplest and most fundamental problems of the infinitesimal calculus [...] and there is no reason to expect any change within this domain. No doubt it is *conceivable* (although one might justifiably judge it unlikely and consign it to a distant future) that a second Cantor should one day give an incontestable arithmetic foundation to new infinitely small numbers, which would then prove themselves of some use in

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[17] Euclid's *Elements*, Book 5, definition 4.
[18] Hegel, *Science of Logic*, 21:239tm.
[19] Hegel, *Science of Logic*, 21:241tm
[20] TN: 'Outside their relation they are pure nullities; but they are to be taken only as moments of the relation, as determinations of the differential coefficient *dx/dy*' (ibid., 21:251).

> mathematics, and which might perhaps open a simple way to infinitesimal calculus.
>
> But as long as no such thing exists [...] we must hold to the idea that one cannot, in any manner, speak of the mathematical – and thus logical – existence of infinitely small numbers, in an identical or analogous sense to that which has been given to the infinitely large.[21]

The strange violence of this text, in spite of the caveats, is symptomatic of an emergent ideological dimension [*est le symptôme d'un affleurement idéologique*]; the history of mathematical analysis is partly entwined with another history, incessantly counteracting it: that of the *repression* [*refoulement*] of infinitesimals. Here Hegel is, to take up an Althusserian expression, merely the philosophical *exploiter* of a remarkably long-lived conjuncture.[22]

At the beginning of the eighteenth century, in his essay *The Analyst*, Berkeley had instituted the merciless prosecution of the foundations of the new calculus, by attacking the weakest link of the theory: the extrapolation of operations, defined for finite magnitudes, to the supposedly 'infinitely small'. We know that Leibniz elided this embarrassing question through recourse, as dubious as it is extravagant, to the metaphysical postulate of pre-established Harmony:

> [...] it will be found that the rules of the finite succeed in the infinite [...] and that [127] *vice versa* the rules of the infinite succeed in the finite [...]: this is because all are governed by reason, and since otherwise there could be neither science nor rule, which would not conform with the nature of the sovereign principle.[23]

It is not difficult to understand why this 'it will be found' no longer satisfied anyone in the eighteenth century. All the more so given that, as Berkeley remarked, different rules applied for actual calculations: infinitesimals did indeed have peculiar operational codes. There was no shame in eventually 'neglecting' the *dx* on the way, when convenient, and the Marquis de l'Hôpital even innocently turns this into a *requirement* [*demande*], right from the beginning of his famous treatise, the first manual of differential calculus: '[we require] that a quantity that is neither increased nor diminished except by another infinitely smaller than it, can be considered as remaining the same'.[24]

But is it possible to maintain that these 'negligences' are 'rules of the finite'? And what is the meaning of this mark *dx*, which both counts and yet doesn't count? How can there be a circumstantial legitimacy to the effacement of an inscription, which one continues to consider as a separable constant?

Take the calculation of the 'difference', as was said at the time, of the product xy, where we already know the difference dx of x, and dy of y, that is to say, the infinitesimals 'associated' with each of these finite magnitudes. I expand $(x + dx)(y + dy)$ yielding: $xy + y\,dx + x\,dy + dx\,dy$. In relation to xy, I thereby have a calculated difference, an 'increase' equal to $y\,dx + x\,dy + dx\,dy$. In order to obtain the classical formula $d(xy) = x\,dy + y\,dx$, I am required to 'neglect' the product $dx\,dy$ of the two

[21] Abraham H. Fraenkel, *Einleitung un die Mengenlehre,* in *Grundlehren der mathematischen Wissenschaften*, vol. 9 (Berlin: Springer, 1928).
[22] In his *Philosophie de l'algèbre* (Paris: PUF, 1962), Jules Vuillemin also denounces any recourse to indivisibles as an intellectual regression: '[…] if one understands by differentials magnitudes at once smaller than our assignable magnitude and nevertheless different from zero, one returns to the precritical epoch of calculus' (523).
[23] Leibniz, *Mémoire de 1701 sur le calcul différentiel*, cited in Abraham Robinson, *Non-Standard Analysis* (Amsterdam: North Holland Publishing Company, 1966), 262-263.
[24] Guillaume de l'Hôpital, *Analyse des infiniment petits pour l'intelligence des lignes courbes* (1696), cited in Robinson, op. cit., 264. De l'Hôpital's book essentially reproduces the ideas of Johann Bernoulli.

infinitesimals. But why now, and not at the very beginning of the calculation? If, in fact, as l'Hôpital says, d*x* d*y* is 'nothing' in relation to *x* d*y* because

$$\frac{dx\,dy}{x\,dy} = \frac{dx}{x}$$

and d*x*, the infinitesimal proper to *x*, is nothing in relation to it, then this is all the more reason why the sum (*x* + d*x*) must be, *from the start*, identified with *x*, such that the calculation no longer makes any sense. For Berkeley, the consecution of the operations *does not hold*, because in the course of the process I change the very principles of this consecution, invoking the rule of negligence only when it happens to suit me.

These objections seem so strong that in truth no attempt was ever made to rebut them, and, as we know, the use of infinitesimals progressively gave way to the 'finitist' notion of the *limit*.

But more essentially, the epistemological nature of the obstacle becomes clearer when one notices that the exclusion of the infinitely small bears on *an infinity-point* relative to the structure of the ordered field [*corps*] of 'magnitudes'. Attempting to think the infinity of the differential, Hegel and all the mathematicians of his day took care above all not to punctualize it: it was this punctualization which classical reason found repugnant.

For an infinitesimal *element* (a 'point') d*x* would then indeed come to occupy the [128] unoccupiable place of the number smaller than all others; the place marked by a variable as site of the impossible. But *there is no* real number that is smaller (or larger) than all others: this is what the theory of continuous positive magnitudes proposes.

However, we will formulate the following *epistemological thesis*: in the history of mathematics, the marking of an infinity-point constitutes the transformation wherein those (ideological) *obstacles* most difficult to reduce are knotted together.

We have seen how, for example, irrational numbers and complex numbers were historically presented as marking an infinity-point ('inexistent' square roots, 'impossible' equations). We know something about the resistance provoked by introduction of the former in Plato's time (the end of the *Theaetetus* is an elaborate discussion of the concept of the minimal element), and by the latter in the period between the Italian algebraists of the sixteenth century and the clarifications proposed by Cauchy.

And as a matter of fact, since it is linked to the forcing of the empty spaces proper to a domain, the introduction of an infinity-point is a modification which must of necessity seem irrational, since in any given theoretical conjuncture rationality is defined precisely by the respect accorded these blank spaces, as the sole guarantors, variably indexed, of *real* difference for the domain. A mathematician like Evariste Galois, whose work is precisely linked to the algebraic theory of infinity-points – the theory of extensions of a basic field [*un corps de base*] – clearly understood that by establishing oneself in the constitutive silence, in the unsaid of a domainial conjuncture, one maintains the chance of producing a decisive reconfiguration. 'It often seems that the same ideas appear at once to many people as a revelation: if one looks for the cause, it is easy to find it in the works of those which preceded, where *these ideas were prescribed unknowingly by their authors*.'[25]

In science as in politics, it is the unperceived or overlooked [*l'inaperçu*] which puts revolution on the agenda.

But the risk taken had to be paid for, in Galois' case, by the uncomprehending ignorance of the academicians. For recasting is a theoretical violence, a subversion.

[25] Évariste Galois, *Écrits et mémoires* (Paris: Gauthier-Villars, 1962), my emphasis.

Lacan's formula, according to which whatever is excluded from the symbolic reappears in the real, can be interpreted here as follows: in certain conditions, the excluded proper to an *already produced* mathematical structure reappears as the instigating mark of a real (historical) process of *production* of a different structure. If we spoke of the hallucinatory form of the infinity-point, as the foreclosed mark which comes back, this is because, arriving at that point where a variable, under the effect of a placed negation, sanctioned the real, the infinity-point declared by a mathematician often provokes accusations of obscurity at best and of madness at worst – and this primarily, as in the case of Galois, from established colleagues, such as Siméon Poisson.

One begins to understand why a mathematics which had *undertaken* the [129] laborious expulsion of infinitesimals subsequently took over, with the interested support of philosophers, the guardianship of the real which this expulsion – baptism of a finally rigorously-founded Analysis – forced it to assume at the beginning of the nineteenth century under the attentive direction of Baron Cauchy.

It is all the more understandable given that the problems raised by Berkeley were truly serious. In their general form, they amounted to the following: What does our definition of the infinity-point imply concerning the extension of algorithms to that impossible term which determines the unoccupiable place where it holds? The surprising inventiveness of the Greeks and the Italian algebraists lay in showing that one can *calculate* with irrationals or imaginary numbers. But in the end the recasting cannot conserve everything. If one closes the real numbers algebraically, no doubt one obtains a macro-field [*surcorps*] (complex numbers) which constitutes their punctual infinitization. But this macro-field is no longer ordered: the structure of order is not valid for the recast domain. If one compactifies the normal topology of these same real numbers by adding a 'point at infinity', the algebraic structure of the field is lost, and so on. Most often, a recasting through the marking of an infinity-point, bound as it is by definition to the possibility of extending *the specific structure of which it is the infinity*, guarantees nothing as to the other procedures defined in the domain, which play no role in the construction of the empty place where the supplementary mark comes.

We know for example that the field of real numbers is *Archimedean*: given two positive numbers a and b, where $a < b$, there always exists another whole number n such that $b < na$.

Now this essential property would not survive the introduction of an infinitely small element dx, defined as the infinity-point of the place that has the property of 'being smaller than all the others'. In fact, for every real positive *finite* number ε, the infinite smallness of dx demands that d$x < \varepsilon$. In particular, for *every* whole number n, d$x < \varepsilon/n$, since ε/n is also a real *finite* number. Consequently, given any positive finite ε and any whole n, for an infinitesimal dx, n d$x < \varepsilon$. One cannot hope to surpass a given finite ε by multiplying the infinitesimal dx by a whole number, however large: the domain of real numbers, recast by the marking of an infinitely small number, is *non-Archimedean*.

Is this an isolated loss? Is it not natural to suspect that the explicit introduction of infinitesimals would wreak such havoc amidst those interleaved structures which constitute the field of real numbers as to leave Analysis paralyzed? This much is clear: in refusing to assign any markable actuality to *dx*, Lagrange following d'Alembert and Hegel following Berkeley *comply with the obstacle*.[26] An epistemological prudence serves here to shore up the repression of a punctual imperceptible [*un imperceptible ponctuel*]. Until just a few years ago, the question appeared to have been resolved: the

[26] TN: 'Lagrange après d'Alembert, Hegel après Berkeley, sont, dans le rejet de toute actualité marquable pour le dx, *selon l'obstacle*.'

nearly-nothing, the infinitely-small, has no mark of its own. The infinitesimal *is not a number*. [130]

IV The Innumerable Numbered

But the infinitesimal *is* a number: a statement which subverts Analysis in the exclusion from which it ended up being born, and which restores, on a new foundation, the inventive innocence of the pioneers of the 'new calculus'.

From a broader perspective, this subversion displaces the uninterrupted effect exerted by Zeno's aporias of continuity and divisibility across several epochs of the concept [*à travers plusieurs époques du concept*]; it rearranges the field of rationality which these aporias governed through the (often mute) imperative that commanded us not to expose ourselves to them.

In the last ten years, the work of Abraham Robinson[27] has established that we can entirely reconstruct classical analysis by 'immersing' the field of real numbers in a non-Archimedean field, by an inaugural marking of an infinity-point – an infinitely large number – and a correlative free recourse to infinitesimal elements.

Aside from finally relieving the secular repression of these concepts, Robinson's discovery administers a convincing proof of the productive capacity of formal thought. In fact, Robinson secures the coherent marking of a large class of infinity-points by exclusive recourse to the theory of formal systems.

Consider the general form of the problem which history bequeathed to us in the form of a refusal: no number exists that is larger than *all* others. That is to say: no number larger than the terms of every strictly increasing *infinite* series. On the other hand, given a *finite* set of numbers, it is very clear that one can always find a number larger than all those in that set. This is even the principle of the *indefiniteness* of the numerical domain, itself subtended by the infinity-support: every finite series can be surpassed. The relation of order transgresses finitude.

Formally, such indefiniteness of a relation (here, that of order) can be expressed as follows: take a formal system S comprising of an infinite set of constants, denoted a_i, (in our example, the marks of numbers), and a binary relation $R\ (x, y)$ where the variables x and y denote the reality of the places to which R distributes the constants (in our example, $R\ (x, y)$ says that $x < y$). Let us suppose that for every finite set of constants $\{a_1, a_2, ..., a_n\}$, it is *coherent* with the axioms of the formal system S to affirm that a constant b exists which maintains, with $a_1, a_2, ..., a_n$, the relation R. [131]

In other words, suppose that for *all finite* sets of constants, the statement:

$$(\exists y)\ [R\ (a_1, y) \cdot \text{and} \cdot R\ (a_2, y) \cdot \text{and} \ldots \cdot \text{and} \cdot R\ (a_n, y)]$$

is coherent with system S.

In this case, relation R structures an indefiniteness over the constants: every finite series $a_1, a_2, ..., a_n$ makes space for the marking of a 'continuation-point ["*point-de-suite*"]' according to R (a majorant or upper bound, in the case where R is the relation of order). To emphasize that the indefiniteness is attached to this marking, we will say that a relation that obeys this condition is transgressive-within-the-finite, or, more simply: *transgressive*.[28]

[27] See the fundamental text, to which we will make constant reference: Abraham Robinson, *Non-Standard Analysis* (Amsterdam: North Holland Publishing Company, 1966). Robinson's discovery is dated autumn 1960. The first publications came in 1961. But the basic idea figured implicitly in Thoralf Skolem's work on non-standard models of arithmetic, work which dates back to 1930-35.
[28] In the English text, Robinson uses the adjective 'concurrent' to characterise such relations.

Now let us consider R_1, R_2, ...,R_n,..., the series of transgressive relations that our system *S* allows (and for simplicity's sake we will suppose that this set is denumerable). We will associate with each of these relations a *supplementary mark*, which does not figure amongst the constants a_1 of the system in its initial form. For the mark associated with R_n we will write ρ_n. We then *adjoin* as new axioms *all* statements of the type R_n (a_i, ρ_n) – that is, all statements affirming that the relation R_n holds between a constant a_i and ρ_n. R_n traverses all the transgressive relations in the series, while a_i successively adopts all possible values among the constants of the system *S*.

In the case of the relation of order over the whole natural numbers, this amounts to associating with < (which is obviously transgressive-within-the-finite) a supplementary mark *i* (which is not the name of a number), and to positing as axioms *all* statements $n < i$, where *n* is a numerical constant. We recognize in *i* an *infinity-point* for the structure of order of the whole natural numbers.

Generally speaking, the new system obtained by the above procedure is *the formal theory of infinity-points for the transgressive-within-the-finite relations of a given system.*

It is important to note that it is a question of a simple *extension* of *S*: all we have done is to *add* a constant and some statements. All the rules and axioms of the initial system remain unchanged, all the theorems of that system are *also* theorems of the theory of infinity-points. In particular, the universally-quantified theorems remain valid, and are thus extended to the 'case' of the supplementary constant (see the appendix to this text).

So, in the formal system of whole numbers, the universal assertion assigning to *every* number *n* a successor $n + 1$ remains true, with the result that to the supplementary constant *i* is assigned a successor $i + 1$. More generally, if we have a theorem of the initial system of the form 'every *x* has the property *P*', elementary logical rules allow us to prove P (a_i) for every constant. In particular, we would then have: 'ρ_n has property *P*'. We have indeed arrived at the conditions in which we can relaunch those algorithms [132] that founded the infinity-point.[29] The *structure* of the initial domain is in certain regards conserved in the recast domain. We will thus call the new system the *transgressive extension* of the initial system.[30]

The key question is evidently that of knowing whether the transgressive extension is a *coherent* system; whether, logically speaking, we have the right to introduce the supplementary axioms required. Will not the addition of *all* the statements of the form R_n (a_i, ρ_n) end up contradicting the fact that the relations R_n are only transgressive within the finite? Because in the system of natural whole numbers it is *false*, for example, that one number can be larger than all the others. Does *infinite* transgression not exceed the logical powers of the formal language adopted?

Pure logic gives us the answer, in the form of a very general theorem, which underlies the whole construction:

If a system is coherent, its transgressive extension is also coherent.[31]

We are thereby authorized in marking an infinity-point for every relation that is transgressive-within-the-finite: this marking conserves the formal coherence and defines a 'non-standard' extension of the structure which is the 'standard' (ordinary) model of the system.

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[29] TN: '*Nous sommes bien dans les conditions de relance des algorithmes, qui fondent l'infini-point*.'
[30] Robinson uses the word 'enlargement'.
[31] This theorem depends upon another, which is fundamental in the theory of formal systems: the theorem of compactness [*compacité*]. This theorem guarantees that a system whose number of axioms is infinite is coherent if all its finite sub-systems (whose number of axioms are finite) are coherent.

Once this has been established, everything runs smoothly. Given the usual theory of real numbers (as a base system), let *R* denote its domain (the 'objects' marked by the constants). The relation of order is obviously transgressive. Take α, the infinity-point relative to this relation: α is 'larger' than every element of *R*: it is *infinitely large*.

Since the universal statements of the initial theory are also valid for α ('return' of algorithms on the infinity-point), and since a sum and its product exist for *every* pair of numbers in *R*, we can define, for example, $\alpha + 1$, $\alpha + \alpha$, or α^n, etc., *all* of which will be infinitely large (larger than every constant of *R*).

We should note, by the way, that the infinity-point α, the *scriptural* instrument of the recasting, retains no particular privilege within the recast domain – a good illustration of the *effacement of the cause* in the apparatus of a structure. In particular, even if α is formally *inscribed* as a unique constant of transgression, it is no more the smallest infinite number than it is the largest – as we have just seen. Thus the number $\alpha - r$, where *r* is any positive number *from the initial domain*, is evidently smaller than α. It nonetheless remains an infinite number. If in fact it were not infinite, it would have to be because it is smaller than a finite number, say *t*. But $\alpha - r < t$ implies that $\alpha < t + r$, [133] which is absurd, α being infinite, and $t + r$, the sum of two finite numbers, being finite. There is in reality an indefinite number of infinite numbers smaller (or larger) than α: the recasting distributes the infinitely large numbers in an open space, both towards the 'bottom' and towards the 'top'. It is in this space that the mark α denotes no assignable, particular position: its operation dissipates it.

Nevertheless, it is clear that every complete *writing* of an infinite number, every trace *effectively* constructed to designate it on the basis of the graphical material of the extension, carries with it at least a mention of α: every writing which combines *only* the constants of the initial system denotes a number of the initial domain, a finite number. The *causality* of the mark α is here, in the domainial effacement of that which it designates, the omnipresence marked for every occupation of a place where only the 'new' infinite numbers can come. The marking of an infinity-point is an operation of the *signifier* as such.

Similarly, the infinitely small is introduced by way of a scriptural combination on the basis of α. We can thereby define $1/\alpha$, since *R* is a field [*corps*], and therefore the statement 'every element has an inverse' is an axiom for *R*. The theorem of the coherence of the extension guarantees the existence of this inverse for the infinitely large element α. Now, this inverse is *infinitesimal* (i.e. infinitely small relative to the constants of *R*).

To illustrate the point, take a real positive *finite* number *a*, as small as you like (a constant of the initial system). It is *always* the case that $a < \alpha$, since α is infinitely large. By dividing the two members of the inequality by the product $a\alpha$ – which is an infinitely large number – we obtain $1/\alpha < 1/a$ for every finite positive *a*, and therefore $1/\alpha < 1/1/a$, since $1/a$ is obviously finite if *a* is finite, i.e. $1/\alpha < a$. Consequently, *whatever* positive finite number *a* we take, $1/a$ is smaller than *a*.

And in turn, this infinitesimal $1/\alpha$ or α^{-1} gives us, by way of an extension of the algorithms, an infinite family of infinitesimals. Specifically, if β is infinitesimal, however large a whole finite number *n* is, $n\beta$ is still infinitesimal. In fact, $\beta < a$ for *every* finite *a* (since β is infinitesimal and *a* finite), and consequently $\beta < a/n$ (a/n remaining finite), and so $n\beta < a$.

In this way we verify that the domain of the extension is non-Archimedean.

Finally, let $R[\alpha]$ be the macro-domain [*surdomaine*] of *R* or *R* recast according to the marking of an infinity-point for the relation of order. It contains, apart from the isomorphic field of real numbers (*R*, denoted by the constants of the initial system), an infinity of infinitely large elements, and of infinitely small elements. [134]

More precisely, let us call *standard* [*conformes*] numbers those marks of $R[\alpha]$ which belong to R, and which are the constants 'from before the recasting'. We can distinguish, amongst the positive numbers of $R[\alpha]$:

— *finite numbers*: numbers included between two standard positive non-null numbers. Naturally, every standard number is finite. But there are other kinds of finite numbers too: for example the sum of a standard number and an infinitesimal is a finite non-standard number.
— *infinite numbers*: numbers larger than every standard number.
— *infinitesimal numbers*: numbers smaller than every standard number (and following convention, we will take zero to be an infinitesimal).

Within this framework, it is possible to provide a very simple definition of something that remained a vague idea during the heroic period of differential calculus: infinite proximity. A number a is infinitely close to a number b if the difference $a - b$ is an infinitesimal number.

It is on this basis that Robinson reconstructs all the fundamental concepts of analysis in a language which, while often reminiscent of that of the Marquis de l'Hôpital, is nevertheless now assured of its systematicity.

Let us remark first of all that in $R[\alpha]$ there exist *whole infinite numbers*: in fact, the transgressive extension of R is *also* an extension of N, the set of whole natural numbers, which is a sub-set of R. Now take a series $s_1, s_2, s_3, \ldots, s_n, \ldots$ of standard numbers. We will say that the standard number l is the *limit* of the series s_n if for every *infinitely large* whole number n, $l - s_n$ is *infinitely small*; the verb 'is' can be substituted for the classical 'tends towards', because to be infinitely large (or small) means: to be an infinite (or infinitesimal) *number*. The concept of convergence is no longer constructed according to the attribution of vanishingness, or tendential properties, but by recourse to *elements* of the defined subsets of $R[\alpha]$.

Thus we find that the principal objection of Hegel – and of Lagrange – to the idea of the limit is subverted by the punctualization of its definition, even as the idea of limit loses its foundational function. For we know that following the decline of infinitely small numbers (a decline marked by d'Alembert's initial clarification of their status), Cauchy, Bolzano and Weierstrass would establish a definitive foundation for differential calculus in the concept of limit: in their eyes, this procedure had the inestimable merit of accepting, thanks to the rationalizing sanction of repression, *only finite marks* in the text. When I say: 'the series s_n has as its limit the number l if, for any positive number ε, there is a whole number M such that $n > M$ entails $|l - s_n| < \varepsilon$', the only numerical constants mentioned (ε, n, M) are all finite. The concept of limit therefore effects a rejection of every infinitesimal mark, and this is precisely why, in the *Encyclopédie*, d'Alembert salutes its positivity:

> It is not at all a question, as one ordinarily says, of infinitely small quantities in *differential* calculus: it is a question only of the limits of finite quantities. Thus the metaphysics of infinite and of infinitely small [135] quantities larger or smaller than each other is totally useless to *differential* calculus. The term *infinitely small* is only used as an abbreviation, to shorten the expression.[32]

Conversely, this positivity, which Hegel also recognized, is for him a failure to express (genuine) infinity. The underlying idea that d*x* marks a proximity, that x 'tends towards' a value x_0, has no speculative meaning for him: 'Approximation or approach [*rapprochement*] is a category that says nothing and renders nothing conceivable: d*x*

[32] Jean d'Alembert, 'Différentiel', *Encyclopédie*, vol. 4 (1754), 985-989.

already has its approach behind it: it is neither nearer, nor further away, and the infinitely near is equivalent to the negation of proximity and of approximation.'[33]

In non-standard analysis, this negation is converted into the numerical *existence* of an infinitesimal, which marks the infinitely small difference. As for the positive ruse of a detour via finite marks, it becomes redundant, for infinite proximity is *numberable* [*chiffrable*]. Both partisans and adversaries of the concept of limit are dismissed, the common terrain of their opposition being defined by their refusal of such a numbering [*chiffrage*].

Similarly, the continuity of a function to the real (standard) point x_0 gives way to statements such as: $f(x)$ is continuous at point x_0, where $a < x < b$, if and only if, for every x infinitely close to x_0 (that is, where $x - x_0$ is infinitesimal), $f(x)$ is infinitely close to $f(x_0)$, which is to say: where $f(x) - f(x_0)$ is infinitesimal.

To define Cauchy's integral, we will divide the interval [*a, b*] into *infinitely numerous* slices (the series x_n of these slices will be indexed on the whole numbers of $R[\alpha]$, which includes infinite whole numbers, so that 'infinitely numerous' has a strict numerical meaning); we will ask that each slice be *infinitely small* (in other words that $x_{n+1} - x_n$ be an infinitesimal number), and so on.

Analysis is indeed shown to be *the site of denumerable infinities*.

Retrospectively, the classical and Hegelian case against infinitesimal quanta is thereby entirely defeated.

No doubt Hegel, or Berkeley, were merely engaging in the spontaneous epistemology of the mathematics of their time. They did not *contradict* these mathematics. But if Berkeley established the fundamental obscurity of Analysis only so as to secure by comparison religion's right to mystery, Hegel in his turn validated the rejection of the infinity-point only in order to run to the aid of a mathematics in search of a foundation so as to bestow upon it the poisoned chalice of 'qualitative' relation. The debasement of multiplicity, the refusal to think the concepts of Analysis according to a logic of marks, however much they may have been fuelled by a confused scientific actuality, are nonetheless *enslaved* to speculative objectives. It is these objectives alone that require the supremacy [136] of quality, and the relative discrediting of algorithmic or *inscribed* thinking: i.e. of structural thinking.

That this retroactive effect has been prepared, throughout the history of philosophy, by a secret and permanent supremacy of the continuous over the discrete, is unequivocally declared by Hegel: 'the variation of variable magnitudes is determined qualitatively and is, consequently, continuous'.[34] Quality and continuity are mutually implicating -- an implication which has weighed upon the very history of the theoretical concepts of differential Calculus, and which has in part governed the censuring of infinitesimals.

Quality, continuity, temporality and negation: the oppressive categories of ideological objectives.

Number, discreteness, space and affirmation: or, better, Mark, Punctuation, Blank Space [*Blanc*] and Cause: the categories of scientific processes.

These are the formal indices of the two 'tendencies' that have been in struggle, according to Lenin, since the beginnings of philosophy. They have been in struggle *within* the discourses themselves, and formative for science's historical choices. A struggle between the materiality of the signifier and the ideality of the Whole.

Within mathematics, infinitesimal traces have been the victims of this struggle: not because they contravened some supposedly formal atemporality, but because a ramified history supported the Reason of an epoch in excluding them, and in not linking [*enchaîner*] the infinite through them.

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[33] Hegel, *Science of Logic*, 21:269tm.
[34] Ibid., 21:278.

That the act and the effect of the infinite should be a question of gaps [*écarts*] and of written supplements, is indeed what no-one wanted to hear, as Cantor's experience showed, two centuries after the founders of the 'new calculus'.[35]

The unforeseen return of infinitesimals, received in a state of renewed astonishment [*stupeur*],[36] even if it arrives too late for Analysis (which is certainly no longer in search of its fundaments or foundations), has the inestimable value of *disintricating* by means of a science that which, in the orchestrated complacency of their rejection, owed less to the necessities of the concept than to those constraining illusions whose salvation required an ideal guarantee. [137]

Appendix

Some may be surprised to see us assert that the axioms of a formal system are 'conserved' for its transgressive extension, whereas, for example, $R[\alpha]$ is non-Archimedean while R is Archimedean. But this perfectly exemplifies the *formal* character of the procedure.

In the initial system, the Archimedeanism is expressed with a *statement* of the type: 'for two numbers a and b such that $a < b$, there always exists a whole number n such that $b < na$'.

We might formalise this statement as follows:

$$(\forall x)\ (\forall y)\ [x < y \rightarrow (\exists n)\ (y < nx)]$$

We say: this formalised statement *is indeed a theorem of* $R[\alpha]$. But of course, the quantified variable 'n' takes its values *from among the whole numbers of* $R[\alpha]$, which includes, as we know, *infinite* whole numbers.

$R[\alpha]$ is not Archimedean, in the sense that for an infinitesimal, there does not exist a *finite n* such that the infinitesimal multiplied by n might surpass a given finite number.

But *the formal statement* of Archimedeanism remains valid, because in multiplying an infinitesimal by a suitable whole *infinite* number, one can indeed surpass every given finite number.

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[35] And as is shown even today, in 1968, by the incredible and grotesque popular success of Georges Antoniadès Métrios' widely-publicized book, *Cantor a tort* [*Cantor is Wrong*] (Puteaux: Sival Presse, 1968) – a risible symptom of the reactionary obstinacy which characterizes the para-mathematical ideologies of the infinite.

[36] Cf. the painful retraction to which Fraenkel very honestly admits in the third edition of his *Abstract Set Theory*, just after a passage devoted to the sterility of the infinitely small: 'Recently, an unexpected use of infinitely small quantities, and particularly a method to found Analysis (calculus) on infinitesimals, have been rendered possible on the basis of a properly non-Archimedean and non-standard extension of the field of real numbers. For this surprising development, the reader... etc.' (*Abstract Set Theory* [Amsterdam: North-Holland, 1966], 125).